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*Lattice and Effective Field  
Theory for  
Cold Fermionic Atoms*

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# QCD and cold atoms: common thread

Investigation of nonperturbative phenomena

## *Effective Field Theory*

Theoretical separation of scales



## *Lattice Calculation*

Monte Carlo evaluation of path integral



## *Rigorous Nonperturbative Results*

# *Cold Fermionic Atoms - Outline*

- **General motivation, specific system**
- **Lattice Field Theory & Monte Carlo calculation**
  - ★ First results -- superfluid/normal phase transition
  - ★ Road to understanding and reducing uncertainties
  - ★ (see M.W., cond-mat/0502372)
- **Symmetries & Low Energy Effective Field Theory**
  - ★ All cold atoms: general coordinate invariance
  - ★ Unitary Fermi gas: scale and conformal invariance
  - ★ (see D.T. Son & M.W., cond-mat/0509786, Ann. Phys. 321, 197 (2006) )
- **Future directions**

# Trapped Atoms are **Versatile!**



## **Atomic theory**

- Feshbach resonances, few-body effects



## **Condensed matter theory**

- Superfluidity, optical lattices and semiconductor physics, very pure systems, interesting phase diagrams



## **Nuclear theory**

- Few body physics, fermion pairing, model of neutron matter



## **Quantum field theory**

- Separation of scales (good for effective field theory), spontaneous symmetry breaking, interesting nonperturbative regime, universality

# The system

Homogeneous gas of 2 identical species of NR fermions

Hamiltonian

$$H = -\frac{1}{2m} \left( \sum_{i=1}^{N_1} \nabla_i^2 + \sum_{j=1}^{N_2} \nabla_j^2 \right) + \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} v(|\vec{r}_i - \vec{r}_j|)$$

Low energy scattering, short range potential

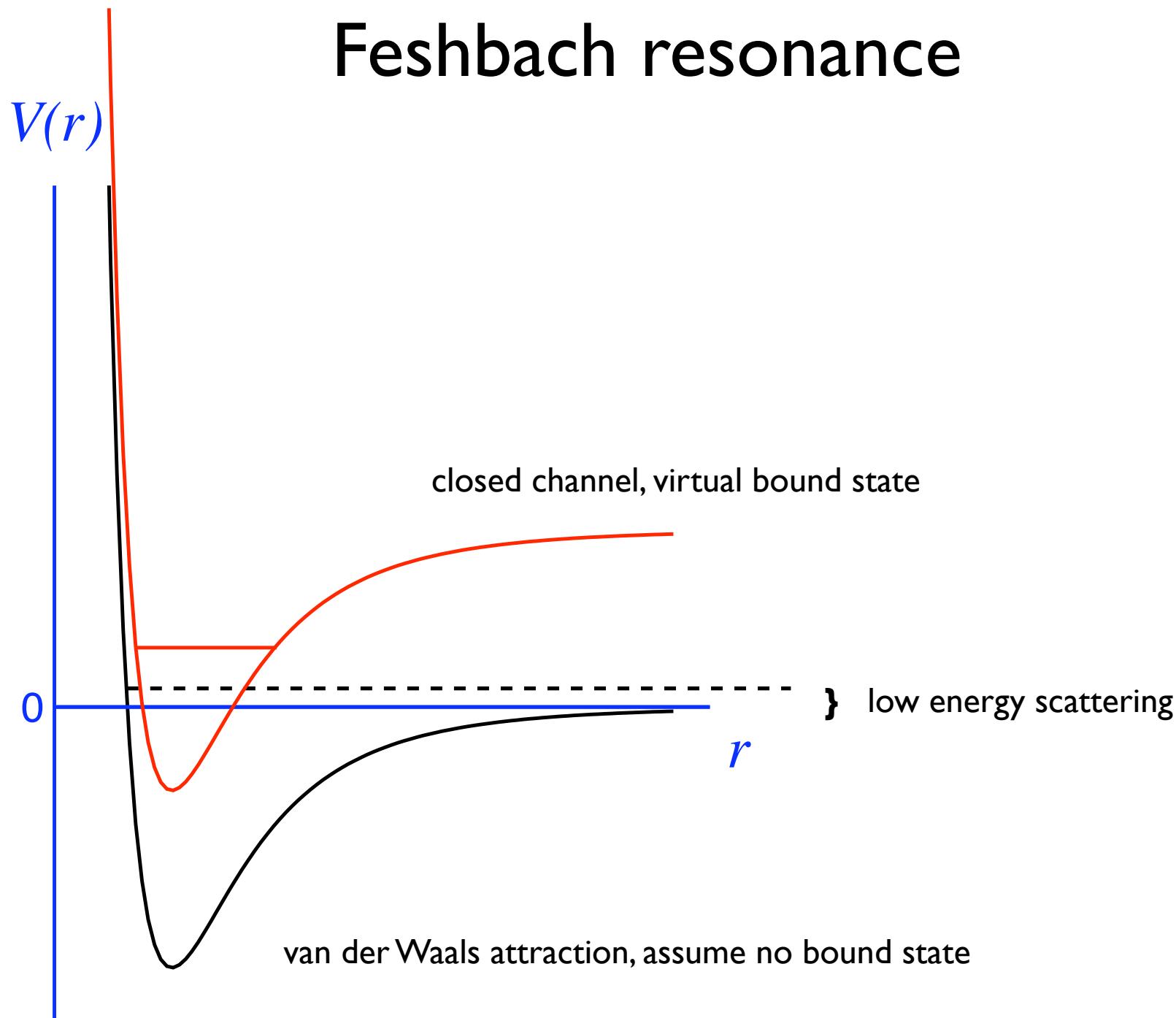
$$\mathcal{A} = \frac{1}{-1/a + \frac{1}{2}k^2 R + \dots - ik}$$

Dilute limit  $R \ll n^{-1/3}$  ( $k_F = (3\pi^2 n)^{1/3}$ )

Relevant physics can be described by scattering length

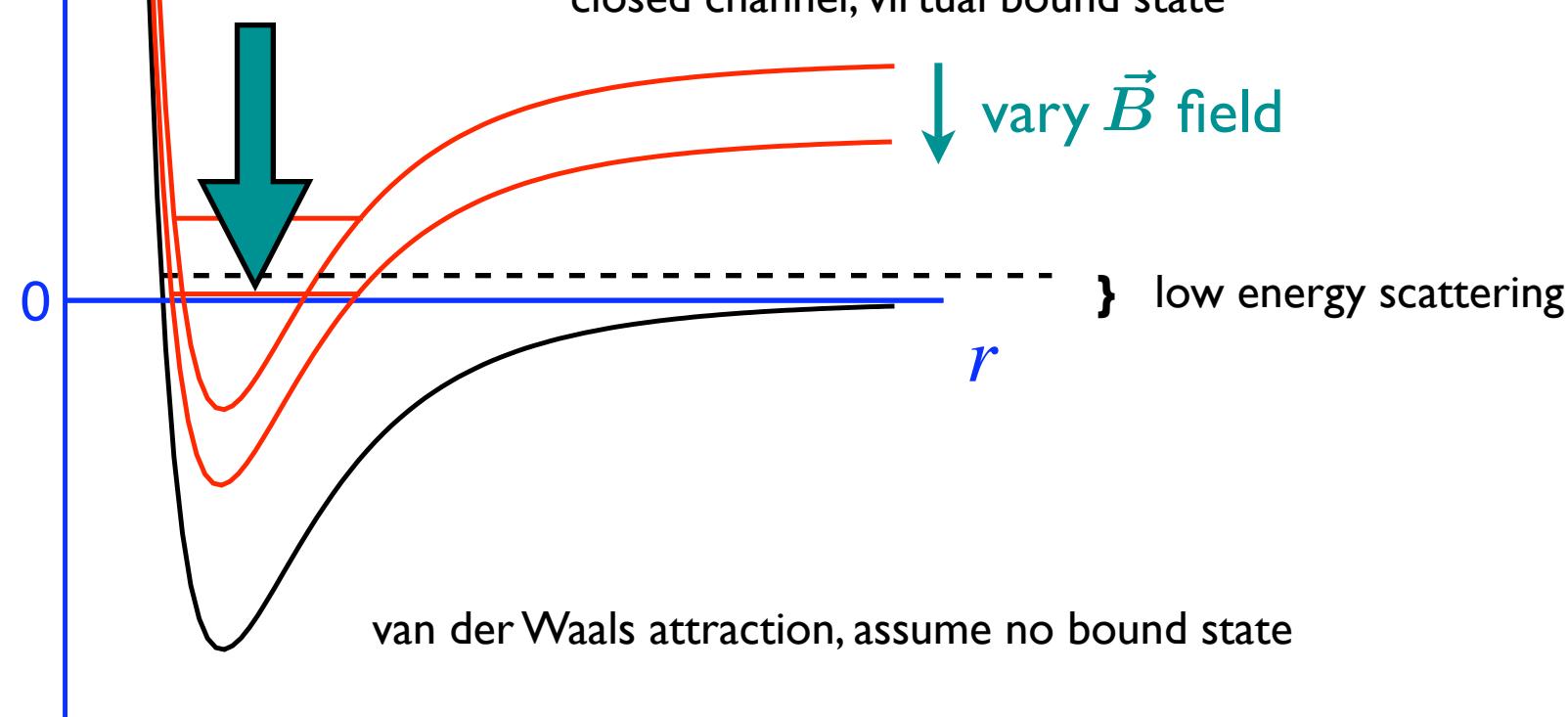
Strongly interacting  $n|a|^3 \gg 1$  “Bertsch problem”

# Feshbach resonance



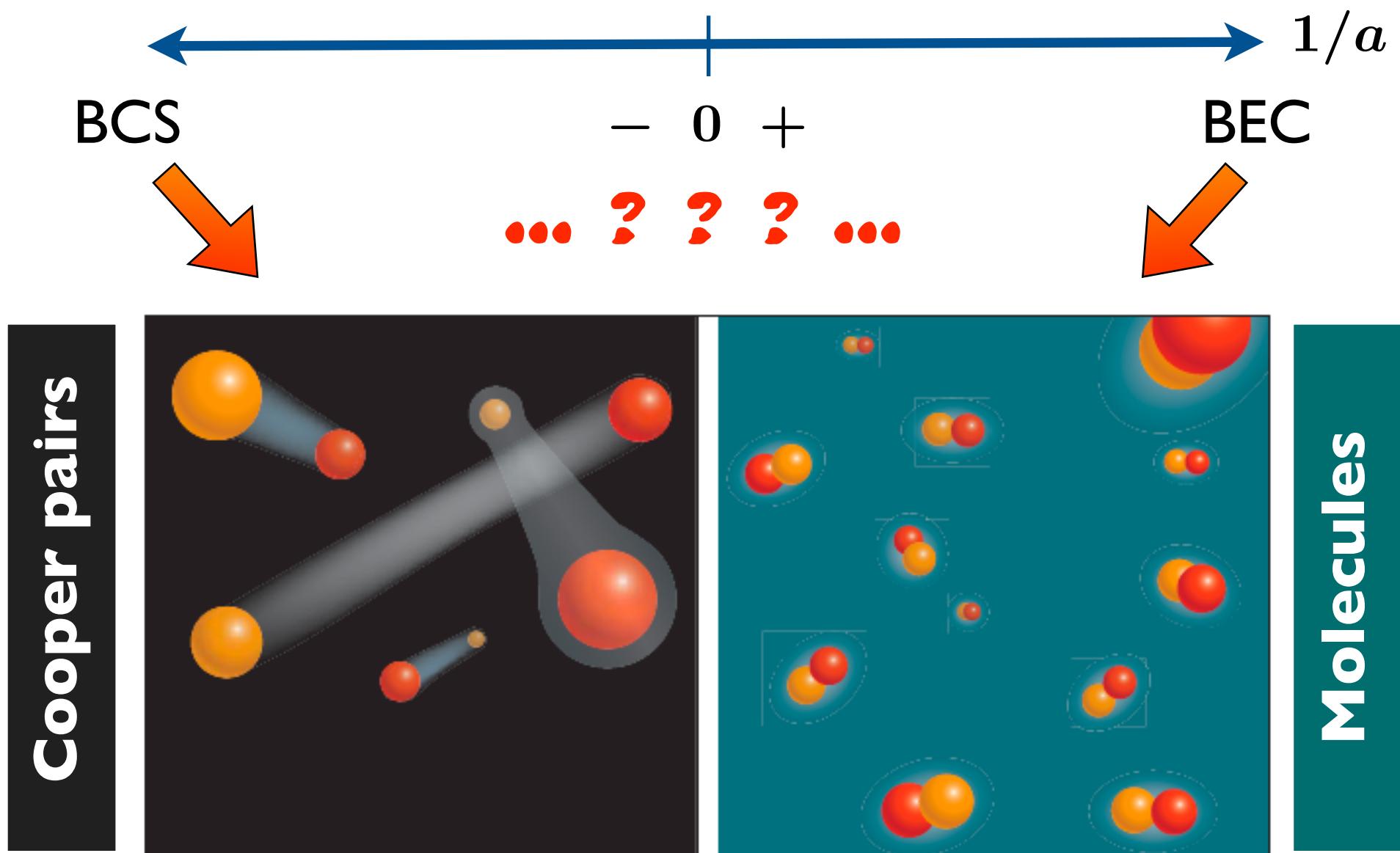
# Feshbach resonance

$V(r)$



$$a = a_{bg} \left( 1 - \frac{\Gamma}{B - B_0} \right)$$

# Fermion pairing at zero temperature

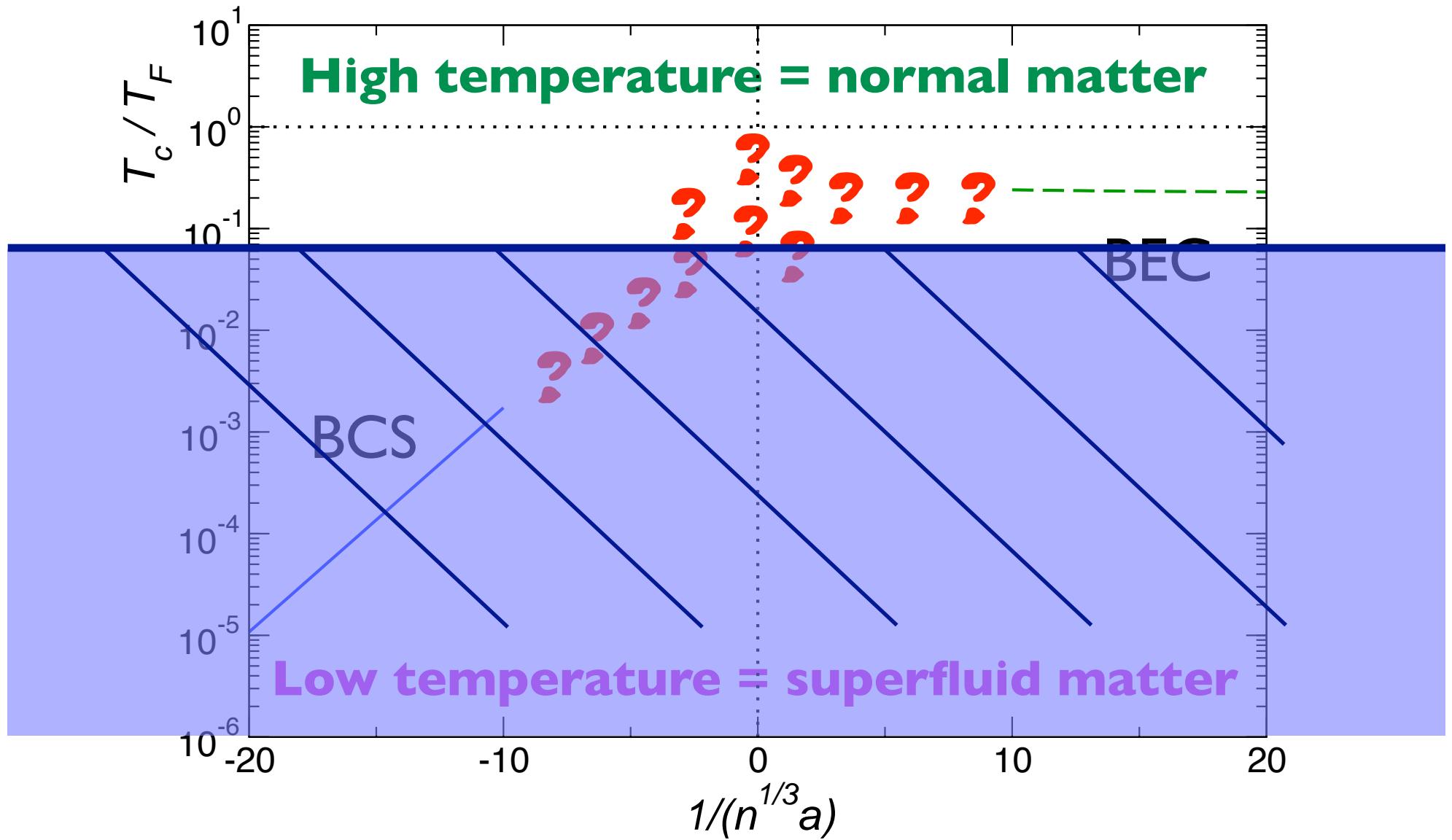


$$-1 \ll an^{1/3} < 0$$

$$0 < an^{1/3} \ll 1$$

Illustration: A. Stonebraker (Science)

# Finite temperature phase transition



# Relevance to nuclear physics

- Many-fermion systems
- Large scattering lengths in  $NN$

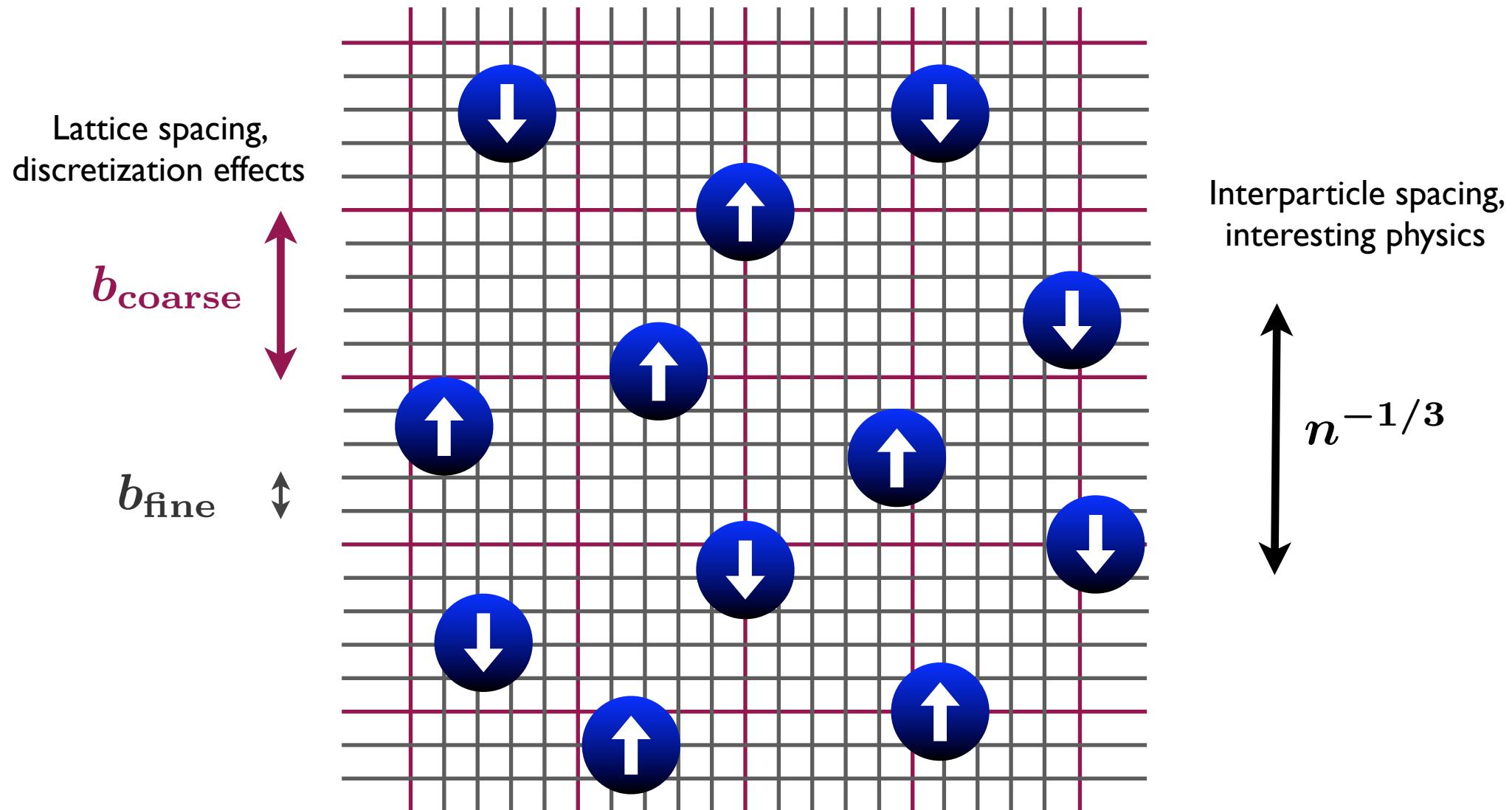
$$\ell_\pi = \hbar/m_\pi c = 1.4 \text{ fm}$$

**$nn$**        $r_s = 2.8 \text{ fm}$        $a_s = -18.5 \text{ fm}$

**$np$**        $^1S_0$        $r_s = 2.75 \text{ fm}$        $a_s = -23.76 \text{ fm}$

**$^3S_1$**        $r_t = 1.76 \text{ fm}$        $a_t = 5.42 \text{ fm}$

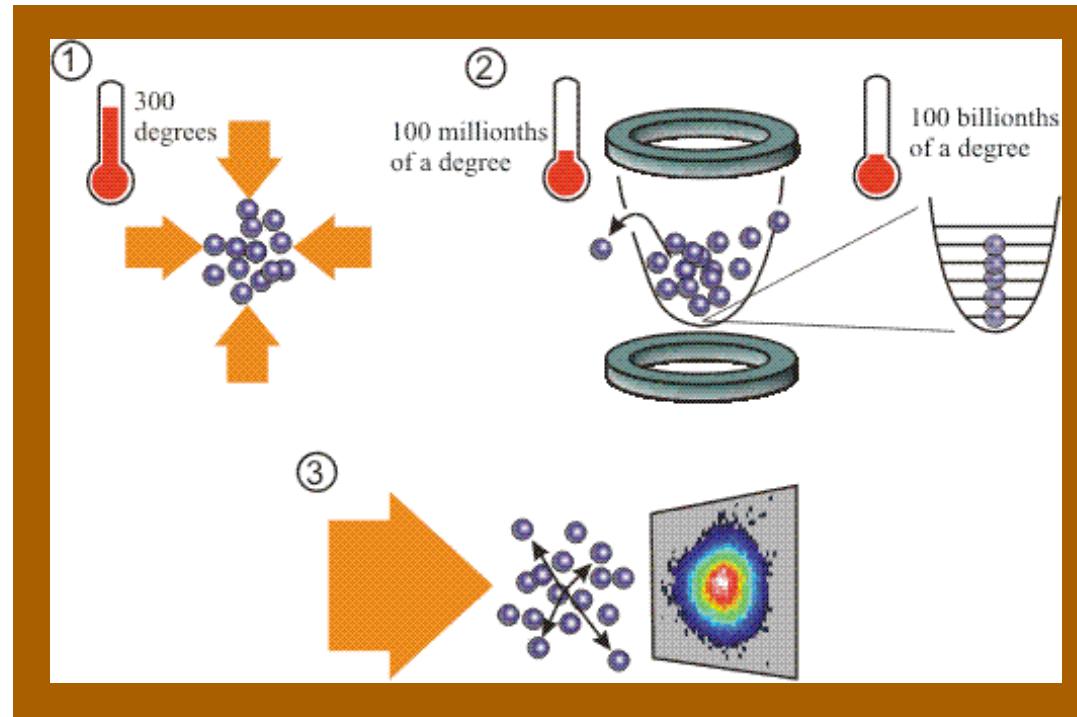
# Physical vs. lattice length scales



The continuum limit is the dilute limit,  $nb^3 \rightarrow 0$

Also worry about finite volume  $b \ll n^{-1/3} \ll L$

# Atomic trap = inhomogeneous matter



Harmonic trap potential  $V(\vec{r}) = \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$

Introduces another physical length scale  $a_{ho} = 1/\sqrt{m\bar{\omega}}$

$b \ll n^{-1/3} \ll a_{ho} \ll L$  **Difficult!**  $(\bar{\omega} \equiv (\omega_x \omega_y \omega_z)^{1/3})$

# Lattice field theory formulation

Write partition function as a path integral

$$\begin{aligned} \mathcal{Z} &= \text{Tr } \exp[-\beta(\hat{H} - \mu \hat{N})] \\ &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left( -\int_0^\beta dx_4 \int d^3x \mathcal{L} \right) \quad \psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\ &\qquad \text{finite 4th dimension (imaginary time)} \end{aligned}$$

$$\mathcal{L} = \bar{\psi} \left( \partial_t - \frac{1}{2} \nabla^2 - \mu \right) \psi + \frac{C_0}{2} (\bar{\psi} \psi)^2$$

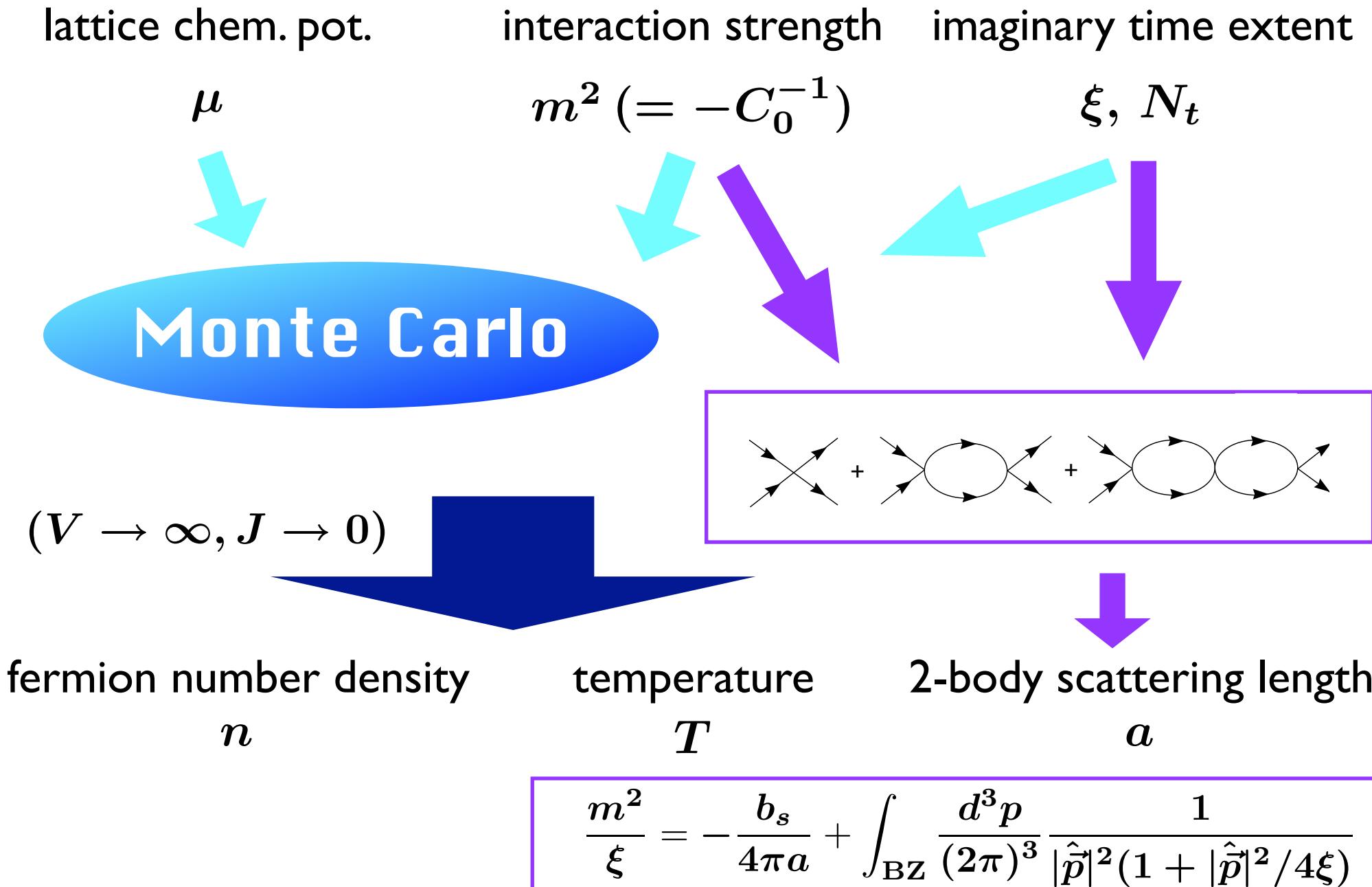
U(1) symmetry: fermion number conservation

$$\psi \rightarrow \exp(i\alpha)\psi, \quad \bar{\psi} \rightarrow \bar{\psi} \exp(-i\alpha)$$

Pair condensation: spontaneous symmetry breaking

$$\langle (\psi^T \sigma_2 \psi + \bar{\psi} \sigma_2 \bar{\psi}^T) + e^{i\theta} (\psi^T \sigma_2 \psi - \bar{\psi} \sigma_2 \bar{\psi}^T) \rangle \neq 0$$

# Tuning of lattice parameters



# Simulation details



Spatial volume

$$V = (8b_s)^3$$



Imaginary time extent

$$N_t = 16$$



fixed chemical potential (so far)

Continuum limit is equivalent to  
 $nb_s^3 \rightarrow 0$

$$\mu b_t = 0.4 \ (\Rightarrow nb_s^3 \approx 0.2 - 0.25)$$



Simulations at several values of

$$(m^2, \xi)$$



3 values of  $J$  (to extrapolate to  $J = 0$ )



Each parameter set, 4000 Monte Carlo steps

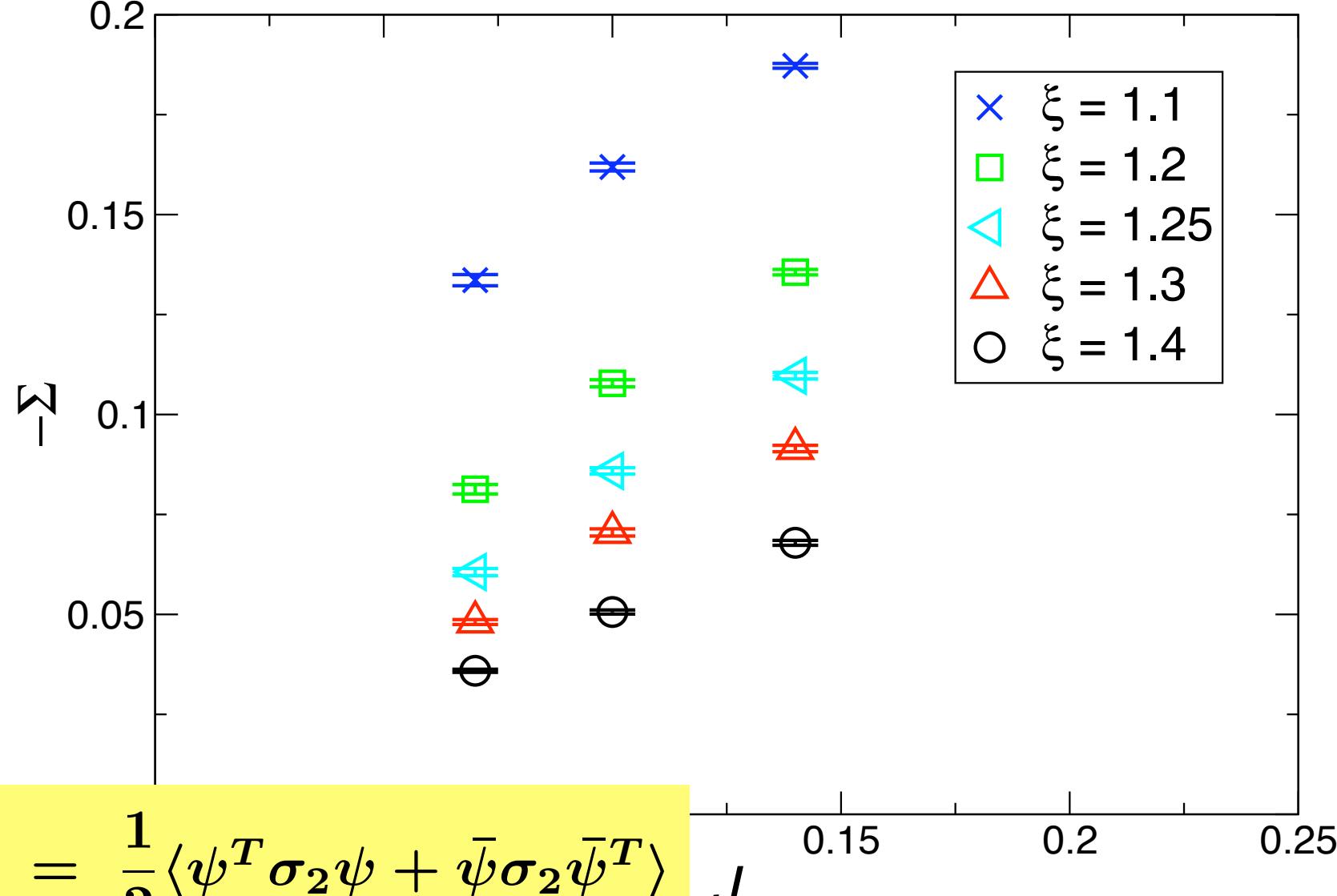


200 independent field configurations (toss first 50)

# Extrapolation to zero external source

Fermion pairing condensate

$$\mu = 0.4, m^2 = 0.175$$

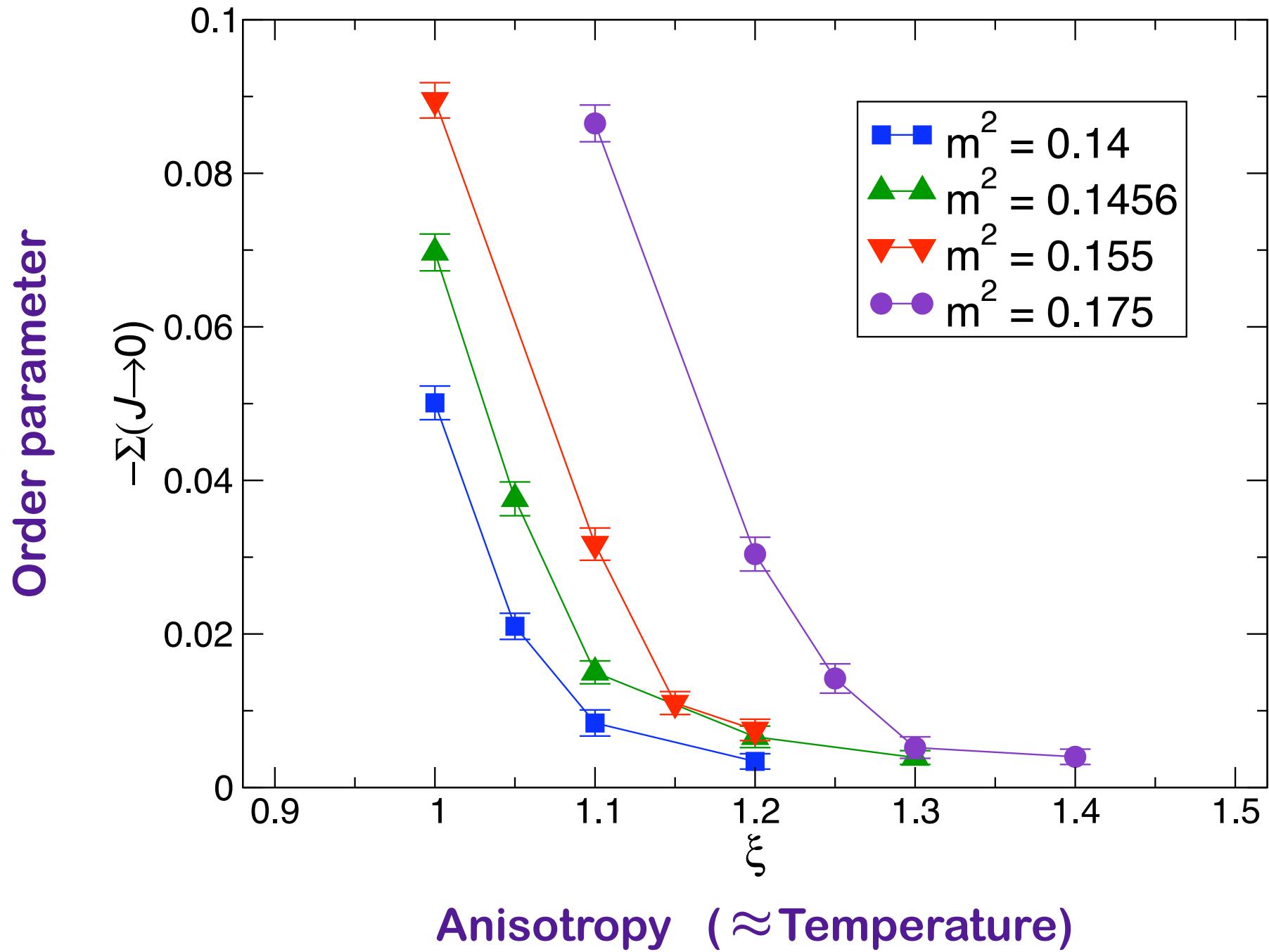


$$\Sigma = \frac{1}{2} \langle \psi^T \sigma_2 \psi + \bar{\psi} \sigma_2 \bar{\psi}^T \rangle$$

$J$

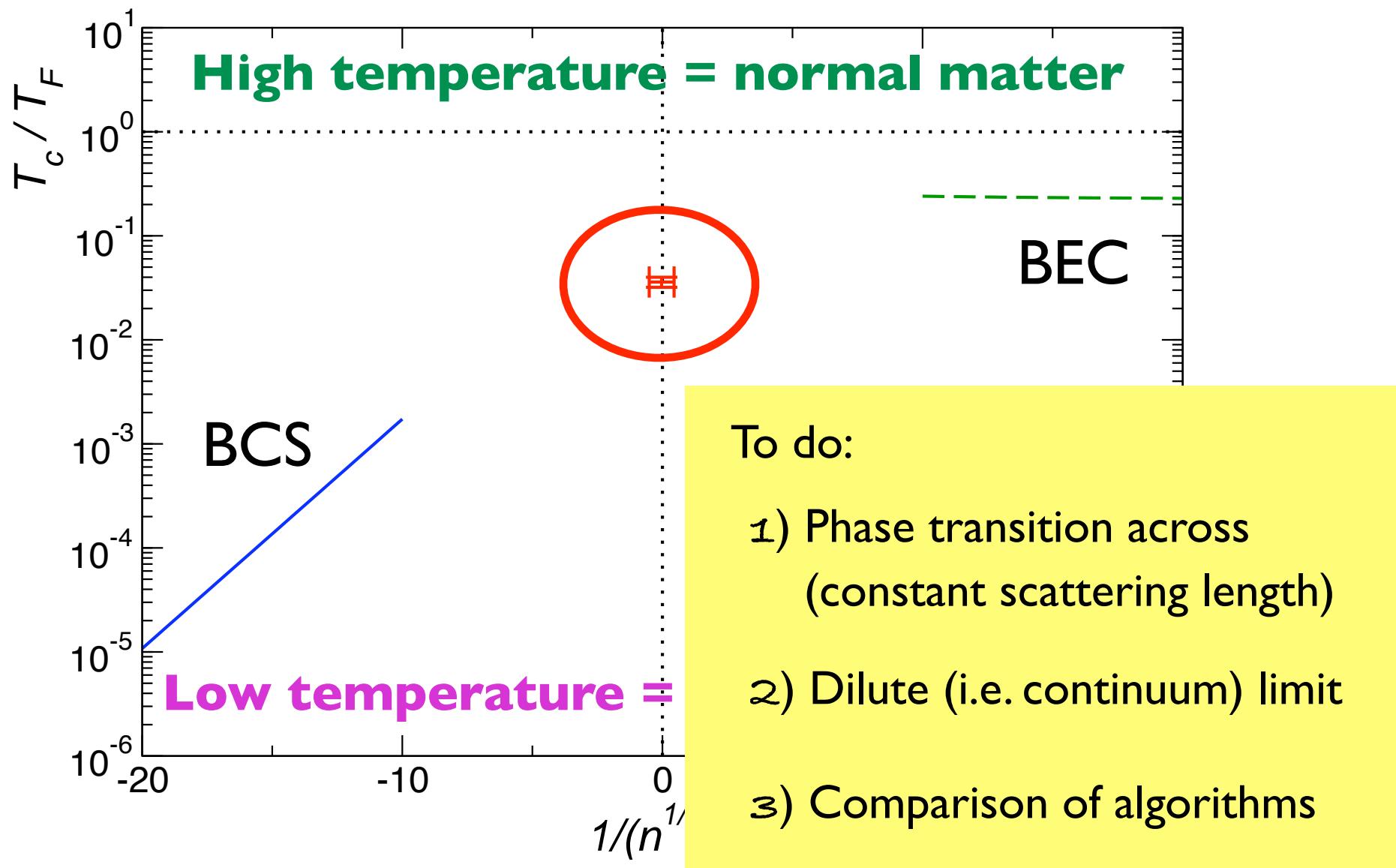
External pairing source strength

# Phase transition



# Exploratory results for critical temperature

M.W., arXiv:cond-mat/0502372



# Other Monte Carlo calculations of $T_c$

- Auxilliary field M.C. (Bulgac, Drut, & Magierski, cm/0505374)
  - Look for shoulders in  $E$  and  $S$  vs.  $T$
  - $T_c/T_F = 0.23(2)$
- Hybrid M.C. (Lee & Schäfer, cm/0509018)
  - $T_c/T_F < 0.14$
- Truncated determinant diagrammatic Monte Carlo  
(Bourovski, et al., cm/0602224)
  - Finite volume analysis, continuum limit
  - $T_c/T_F = 0.152(7)$

# Low energy dynamics -- Overview

- At very low temperatures, the only excitations are **phonons**

$$\text{thermal wavelength } \sqrt{\frac{2\pi}{mT}} \gg \frac{v_F}{\Delta_0} \text{ coherence/healing length}$$

- Leading order behavior: superfluid hydrodynamics (in bulk)  
Thomas-Fermi theory (in traps)
- Use **symmetries** of fermion Lagrangian to construct phonon effective Lagrangian, **beyond leading order**
- Predictions for experiment **and Monte Carlo calculations**

# Introducing external sources

$$\mathcal{L} = \frac{i}{2}\psi^\dagger \overleftrightarrow{D}_t \psi - \frac{g^{ij}}{2m} D_i \psi^\dagger D_j \psi + q_0 \psi^\dagger \psi \sigma - \frac{1}{2} g^{ij} \partial_i \sigma \partial_j \sigma - \frac{\sigma^2}{2r_0^2}$$

U(1) gauge field  $D_\mu \psi = (\partial_\mu + iA_\mu)\psi$       3D metric tensor  $g^{ij}$

$$\exp\{iW[A_0, A_i, g_{ij}]\} = \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \exp\{iS[\psi, \psi^\dagger, A_0, A_i, g_{ij}]\}$$

number density

$$n = -\frac{\delta W}{\delta A_0}$$

momentum density

$$T_{0k} = -mg_{ik}\frac{\delta W}{\delta A_i} + A_k \frac{\delta W}{\delta A_0}$$

number current

$$j^k = -\frac{\delta W}{\delta A_k}$$

stress tensor

$$T_k^i = 2g_{kj}\frac{\delta W}{\delta g_{ij}} - \delta_k^i W + A_k \frac{\delta W}{\delta A_i}$$

# The Logic

- Arguments for validity of this microscopic theory
- Gauge field, spatial curvature are sources for fermion number density, current density, and stress tensor -- spurion analysis
- Nonrelativistic limit of relativistic theory
- Hypothesize this is correct microscopic Lagrangian
- Testable hypothesis, has observable consequences

# Symmetry breaking and the phonon

U(1) symmetry, conservation of particle number

$$\psi \rightarrow e^{i\alpha} \psi, \quad \psi^\dagger \rightarrow \psi^\dagger e^{-i\alpha}$$

Superfluidity from spontaneous symmetry breaking

$$\langle \psi \psi \rangle = |\langle \psi \psi \rangle| e^{-2i\varphi} \neq 0$$

$\varphi(t, \vec{x})$  is the corresponding Goldstone mode, the phonon

Absorb chemical potential

$$\theta(t, \vec{x}) \equiv \mu t - \varphi(t, \vec{x})$$

# General coordinate invariance

$$x^i \rightarrow x^i + \xi^i(t, \vec{x})$$

$$\delta\psi = i\alpha\psi - \xi^k \partial_k \psi$$

$$\delta A_0 = -\dot{\alpha} - \xi^k \partial_k A_0 - A_k \dot{\xi}^k$$

$$\delta A_i = -\partial_i \alpha - \xi^k \partial_k A_i - A_k \partial_i \xi^k + m g_{ik} \dot{\xi}^k$$

$$\delta g_{ij} = -\xi^k \partial_k g_{ij} - g_{ik} \partial_j \xi^k - g_{kj} \partial_i \xi^k$$

$$\delta\theta = \alpha - \xi^k \partial_k \theta$$

$$\delta(D_t \theta) = -\xi^k \partial_k D_t \theta - \dot{\xi}^k D_k \theta$$

$$\delta(g^{ij} D_i \theta D_j \theta) = -\xi^k \partial_k (g^{ij} D_i \theta D_j \theta) - 2m \dot{\xi}^k D_k \theta$$

$$\mathcal{L}_{\text{eff}}^{(0)} = P(X) \quad X \equiv D_t \theta - \frac{g^{ij}}{2m} D_i \theta D_j \theta$$

**Note: GCI holds for cold atoms in general!**

# Galilean invariance

... is a special case of general coordinate invariance

$$g^{ij} = \delta^{ij} \quad \alpha = m\vec{v} \cdot \vec{x} \quad \xi^k = v^k t$$

$$\begin{aligned}\psi(t, \vec{x}) &\rightarrow \psi'(t, \vec{x}) = e^{im\vec{v} \cdot \vec{x}} \psi(t, \vec{x} - \vec{v}t) \\ A_0(t, \vec{x}) &\rightarrow A'_0(t, \vec{x}) = A_0(t, \vec{x} - \vec{v}t) - v^k A_k(t, \vec{x} - \vec{v}t) \\ A_i(t, \vec{x}) &\rightarrow A'_i(t, \vec{x}) = A_i(t, \vec{x} - \vec{v}t)\end{aligned}$$

We will find that at NLO, general coordinate invariance is **more restrictive** than Galilean invariance

# Leading order phonon Lagrangian

$$\mathcal{L}_{\text{eff}}^{(0)} = P(X) \quad X \equiv D_t \theta - \frac{g^{ij}}{2m} D_i \theta D_j \theta$$

$P(X)$  is the same function as the pressure  $P(\mu)$

$$n = -\frac{\partial \mathcal{L}}{\partial A_0} = -\frac{\partial \mathcal{L}}{\partial X} \frac{\partial X}{\partial A_0} = \frac{\partial P}{\partial X} \quad n = \frac{\partial P}{\partial \mu}$$

Unitary Fermi gas: scale invariance and dimensional analysis

$$P(X) = c_0 m^{3/2} X^{5/2}$$

Energy per particle

$$\varepsilon(n) = \frac{3}{5} \left( \frac{2n}{5c_0} \right)^{2/3} \frac{n}{m} = \xi \frac{3}{5} \frac{(3\pi^2 n)^{2/3}}{2m} n$$

# Phonon Lagrangian

General coordinate invariance yields the combination

$$\mathcal{L} = P(X) \quad X = D_t \theta - \frac{(D_i \theta)^2}{2m}$$

$$+ \partial_i X \partial_i X f_1(X) + (\partial_i D_i \theta)^2 f_2(X)$$

$$+ \left( -m \partial_i E_i + \frac{1}{4} F_{ij} F_{ij} - \partial_i F_{ij} D_j \theta \right) f_3(X)$$

$$(\mathcal{L}_4 = R_{3D} f_4(X))$$

3 unknown fns

Galilean invariant operators which don't satisfy GCI

$$F_{ij} F_{ij}, \quad m \partial_i E_i + \partial_i F_{ij} D_j \theta$$

$$m^2 E_i^2 + 2m E_i F_{ik} D_k \theta + F_{ij} F_{ik} D_j \theta D_k \theta$$

# Scale invariance

Unitary Fermi gas

$$t \rightarrow t' = \lambda^{-1}t \quad \vec{x} \rightarrow \vec{x}' = \lambda^{-1/2}\vec{x}$$

(really a coordinate transformation)

$$f_1(X) = c_1 m^{1/2} X^{-1/2}$$

$$f_2(X) = c_2 m^{-1/2} X^{1/2}$$

$$f_3(X) = c_3 m^{-1/2} X^{1/2}$$

*Functions become known, up to multiplicative constants*

# Conformal invariance

More general reparameterization of time

$$t \rightarrow t' = t + \beta(t)$$

scale transformation:  $\beta(t) = bt$

$$\delta\psi = -\beta\dot{\psi} - \frac{3}{4}\dot{\beta}\psi$$

$$\delta A_0 = -\beta\dot{A}_0 - \dot{\beta}A_0$$

$$\delta g_{ij} = -\beta\dot{g}_{ij} + \dot{\beta}g_{ij}$$

$$\delta A_i = -\beta\dot{A}_i$$

Invariance of Yukawa-like interaction

$$[\psi] = \frac{3}{4} = [\sigma] \quad [q_0] = \frac{1}{4} \quad \& \quad [r_0] = -\frac{1}{2} \Rightarrow [q_0^2 r_0] = 0$$

Further constrains NLO EFT

$$c_3 = -9 c_2$$

# Phonon Lagrangian to NLO

General coordinate invariance yields the combination

$$\begin{aligned}\mathcal{L} &= P(X) & X &= D_t \theta - \frac{(D_i \theta)^2}{2m} \\ &+ \partial_i X \partial_i X f_1(X) & &+ (\partial_i D_i \theta)^2 f_2(X) \\ &+ \left( -m \partial_i E_i + \frac{1}{4} F_{ij} F_{ij} - \partial_i F_{ij} D_j \theta \right) f_3(X)\end{aligned}$$

all gases

Conformal invariance tells us the functional form

$$\begin{aligned}\mathcal{L} &= c_0 m^{3/2} X^{5/2} + c_1 \sqrt{m} (\partial_i X \partial_i X) X^{-1/2} \\ &+ \frac{c_2}{\sqrt{m}} \left[ (\partial_i D_i \theta)^2 + 9m \partial_i E_i - \frac{9}{4} F_{ij} F_{ij} + 9 \partial_i F_{ij} D_j \theta \right] X^{1/2}\end{aligned}$$

unitary Fermi gas

# Applying the EFT

## Equation of state

$$\frac{E}{N} = \xi \frac{E_{\text{free}}}{N} \quad c_0 = \frac{2^{5/2}}{15\pi^2 \xi^{3/2}}$$

## Dispersion relation

$$\omega(q) = \sqrt{\frac{\xi}{3}} v_F q \left[ 1 - \pi^2 \sqrt{2\xi} \left( c_1 + \frac{3}{2} c_2 \right) \frac{q^2}{k_F^2} \right] + O(q^5 \ln q)$$

## Static density response function

$$\chi(q) = -\frac{m k_F}{\pi^2 \xi} \left[ 1 + 2\pi^2 \sqrt{2\xi} \left( c_1 - \frac{9}{2} c_2 \right) \frac{q^2}{k_F^2} \right] + O(q^4 \ln q)$$

## Static transverse response function

$$\chi^T(q) = -9 c_2 \sqrt{\frac{\xi}{2}} v_F q^2 + O(q^4 \ln q)$$

# Prediction

- ➊ Eliminate unknown constants to obtain a prediction

$$\omega(q) = \sqrt{\frac{\xi}{3}} v_F q \left[ 1 - \frac{\chi(q) - \chi(0)}{2\chi(0)} + \frac{4\pi^2}{3} \frac{\chi^T(q)}{v_F k_F^2} \right]$$

- ➋ Not testable by experiment in near future
- ⌾ Challenge for Monte Carlo methods
- ➌ Make more predictions

# *Current and future effort - atoms*

- **Lattice field theory** -- Very sound theoretical approach
  - No ground state input necessary
  - No sign problem (therefore, no fixed-node approximation, etc.)
  - Errors are systematically reducible
  - More precision by determining lines of constant physics in lattice parameter space
- **Effective field theory**
  - Utilize all possible symmetries
  - Systematically include corrections to superfluid hydrodynamics and Thomas-Fermi theory
  - Finite volume, finite source ( $J$ ) EFT in progress w/ J.-W. Chen

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# Relativistic Root of Gen. Coord. Inv.

Relativistic, free boson action in curved spacetime

$$S = - \int d^4x \sqrt{-g} (g^{\mu\nu} \partial_\mu \Psi^* \partial_\nu \Psi + m^2 c^2 \Psi^* \Psi)$$

Nonrelativistic limit  $c \rightarrow \infty$

$$\Psi = \frac{e^{-imc^2 t}}{\sqrt{2mc}} \psi \quad g_{\mu\nu} = \begin{pmatrix} -1 - \frac{2A_0}{mc^2} & -\frac{A_i}{mc} \\ -\frac{A_i}{mc} & g_{ij} \end{pmatrix}$$

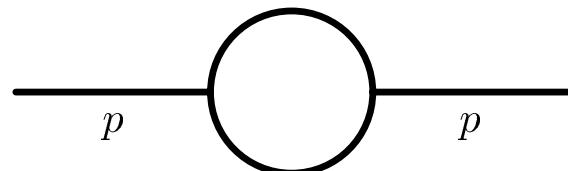
$$S = \int dt d^3x \sqrt{g} \left[ \frac{i}{2} \psi^\dagger \overleftrightarrow{\partial}_t \psi - A_0 \psi^\dagger \psi - \frac{g^{ij}}{2m} (\partial_i \psi^\dagger - i A_i \psi^\dagger)(\partial_j \psi + i A_j \psi) \right]$$

Nonrelativistic GC transformations follow from relativistic GCI

# Loops enter at NNLO

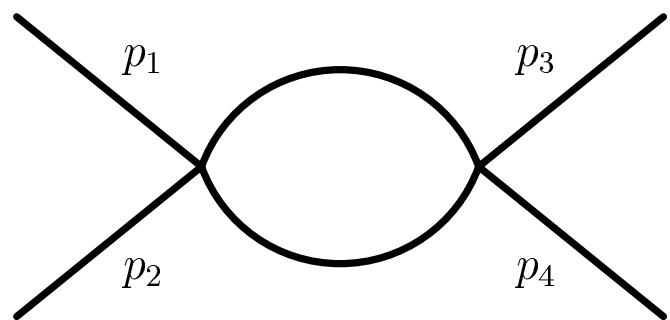
Rescale field to obtain canonical normalization for K.E.

$$\mathcal{L} \sim (\partial\varphi)^2 + \frac{\#}{\mu^2}(\partial_0\varphi)(\partial_i\varphi)^2 + \frac{\#}{\mu^4}(\partial\phi)^4 + \dots$$



$$\frac{p^4}{\mu^4} p^2 \ln(p/\mu)$$

vs. tree  $p^2$



$$\frac{p^8}{\mu^8} \ln(p/\mu)$$

vs. tree  $\frac{p^4}{\mu^4}$

Loops suppressed vs. tree by

$$\frac{p^4}{\mu^4} \ln(p/\mu)$$